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# Static plane-symmetric solutions in Brans–Dicke and Sen–Dunn theories of gravitation

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**Abstract.** Vacuum field equations for the static plane-symmetric metric are obtained in Brans–Dicke and Sen–Dunn scalar–tensor theories of gravitation. Closed form exact solutions to the field equations are presented and studied in both theories.

## 1. Introduction

Brans and Dicke (1961) have formulated a scalar–tensor theory of gravitation in which the tensor field alone is geometrized and the scalar field is alien to the geometry. Recently, Sen and Dunn (1971) proposed a new scalar–tensor theory of gravitation in a modified Riemannian manifold in which both the scalar and tensor fields have intrinsic geometrical significance. The scalar field, in this theory, is characterized by the function $x^0 = x^0(x^i)$ where $x^i$ are the coordinates in the four-dimensional Lyra manifold and the tensor field is identified with the metric tensor $g_{ij}$ of the manifold.

The field equations given by Sen and Dunn for the combined scalar and tensor fields are

$$R_{ij} - \tfrac{1}{2}g_{ij}R = 8\pi G(x^0)^{-2}T_{ij} + \omega(x^0)^{-2}(x^0_{,i}x^0_{,j} - \tfrac{1}{2}g_{ij}x^0_{,k}x^{0,k}) \tag{1}$$

where $\omega = \frac{3}{2}$, $T_{ij}$ is the energy–momentum tensor of the field and $R$ is the usual Riemann curvature scalar. It was pointed out that these equations are identical to the Brans–Dicke equations, namely,

$$\begin{aligned} R_{ij} - \tfrac{1}{2}g_{ij}R &= 8\pi\phi^{-1}T_{ij} + \omega\phi^{-2}(\phi_{,i}\phi_{,j} - \tfrac{1}{2}g_{ij}\phi_{,k}\phi^{,k}) + \phi^{-1}(\phi_{,i;j} - g_{ij}\Box\phi) \\ \Box\phi &= 8\pi T(3+2\omega)^{-1} \end{aligned} \tag{2}$$

if the scalar function satisfied the condition

$$\phi_{,i;j} - g_{ij}\Box\phi = 0$$

and $\omega = \frac{3}{2}$. However, the gravitational constant must be re-defined. Brans (1962) and Mahanta and Reddy (1971, 1972, 1974) gave spherically symmetric static solutions in the Brans–Dicke theory of gravitation, while Sen and Dunn (1971), Halford (1972) and Reddy (1976) have obtained the same in the Sen–Dunn theory.

This paper presents exact static plane-symmetric solutions to the vacuum field equations of Brans–Dicke and Sen–Dunn scalar–tensor theories of gravitation. It is observed that these solutions are formally similar to static plane-symmetric zero mass meson solutions of Einstein's equations given by Patel (1975). Although the solutions we provide are rather highly specialized, they represent a new line of approach towards obtaining plane wave solutions to the Brans–Dicke theory.

## 2. Field equations and solutions

We consider the plane-symmetric line element

$$ds^2 = e^{2\alpha}(dt^2 - dx^2) - e^{2\beta}(dy^2 + dz^2) \tag{3}$$

where $\alpha$ and $\beta$ are functions of $x$ only. For this space–time the non-vanishing components of the Ricci tensor $R^i_j$ are given by

$$\begin{aligned} R^1_1 &= -e^{-2\alpha}(\alpha'' + 2\beta'' + 2\beta'^2 - 2\alpha'\beta') \\ R^2_2 = R^3_3 &= -e^{-2\alpha}(\beta'' + 2\beta'^2) \\ R^4_4 &= -e^{-2\alpha}(\alpha'' + 2\alpha'\beta'). \end{aligned} \tag{4}$$

Here we have numbered the coordinates $x, y, z, t$ as $x^1, x^2, x^3, x^4$. A superscript prime indicates differentiation with respect to $x$.

### 2.1. *Brans–Dicke theory*

Taking $\phi$ as a function of $x$ only and using (4) and (3) in (2) the Brans–Dicke field equations, in vacuum, can be written as

$$\begin{aligned} &\phi(\beta'^2 + 2\alpha'\beta') = -(\omega\phi'^2/2\phi + \phi'' - \alpha'\phi') \\ &\phi(\beta'' + \beta'^2 + \alpha'') = (\omega\phi'^2/2\phi - \beta'\phi') \\ &\phi(2\beta'' + 3\beta'^2 - 2\alpha'\beta') = (\omega\phi'^2/2\phi - \alpha'\phi') \\ &\phi'' + 2\beta'\phi' = 0. \end{aligned} \tag{5}$$

It can be verified easily that when the scalar field $\phi$ is constant, the field equations (5) yield a solution which describes an empty space–time, discussed by Taub (1951), in Einstein's theory. When $\phi$ is not a constant, but is a function of $x$ only, the Brans–Dicke field equations (5) admit the closed form exact solution given by

$$\begin{aligned} 2\beta &= \ln(\mu \pm 2\lambda x) \\ 2\alpha &= a \ln(\mu \pm 2\lambda x) + \alpha_0 \\ \phi &= \pm\left(\frac{A}{2\lambda}\right) \ln(\mu \pm 2\lambda x) + \phi_0 \end{aligned} \tag{6}$$

and the constant $\lambda$ is given by

$$\lambda^2 = e^{4\beta}\beta'^2 = \frac{Q \pm (Q^2 - 4PR)^{1/2}}{2P} \tag{7}$$

where

$$P = 2(1 + 2a) \qquad Q = 2C(a + 2) \qquad R = \omega C^2 \tag{8}$$

and $\alpha_0$, $\phi_0$, $\mu$, $A$, $C$ and $a$ are constants of integration. Thus (3) and (6) along with (7) and (8) constitute static plane-symmetric solutions for the Brans–Dicke theory of gravitation. In the solutions (6) $\mu$ can be set equal to zero by a simple change of origin for $x$, the $\pm$ signs can be fixed by considering the signature of the metric and $\alpha_0$ can be set equal to unity by a change of scale of $x$ and $t$. These solutions are formally similar to the static plane-symmetric solutions of the field equations for zero mass meson fields obtained by Patel (1975). Also it can be seen that when $A = 0$, $\phi = \phi_0$ the solutions reduce to the general relativity solutions given by Taub (1951).

### 2.2. *Sen–Dunn theory*

Taking the scalar field $x^0$ as a function of $x$ only and using (4) and (3) in (1) the Senn–Dunn field equations, in vacuum, are

$$\begin{aligned} &2\beta'^2 + 4\alpha'\beta' = -\omega h'^2 \\ &2\beta'' + 2\beta'^2 + 2\alpha'' = \omega h'^2 \\ &4\beta'' + 6\beta'^2 - 4\alpha'\beta' = \omega h'^2 \end{aligned} \tag{9}$$

where we have put $x^0 = e^h$.

It can be pointed out that with $T_{ij} = 0$, then Sen–Dunn field equations are identical to the Einstein equation for a zero mass scalar field and so the solutions of Patel (1975) of the latter system can be taken over directly to the Sen–Dunn theory. But, for completeness, we write the solutions of the field equations (9) as

$$\begin{aligned} &2\beta = \ln(b_1 x + b_2) \\ &\alpha = (b_3/b_1)\ln(b_1 x + b_2) + b_4 \\ &x^0 = b_5(b_1 x + b_2)^{\lambda_1} \end{aligned} \tag{10}$$

where the $b_i$ are constants of integration and $\lambda_1$ is given by

$$\lambda_1^2 = -\frac{1}{2\omega}\left(1 + \frac{4b_3}{b_1}\right). \tag{11}$$

Thus (3) and (10) along with (11) constitute plane-symmetric static solutions in the Sen–Dunn theory. It is interesting to note in this case that when $\omega \to \infty$, $x^0 = b_5 =$ constant and the solution (10) goes over to the general relativity solutions given by Taub (1951).

## 3. Conclusions

Closed form exact solutions to the field equations of scalar–tensor theories formulated by Brans and Dicke and Sen and Dunn are presented for the static plane-symmetric metric which are formally similar to the plane-symmetric static solutions of Patel (1975) for zero mass meson fields. Although the solutions we provide are rather highly specialized, they represent a new line of approach towards obtaining plane wave solutions to the Brans–Dicke theory.

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## References

Brans C 1962 *Phys. Rev.* **125** 2194–201
Brans C and Dicke R H 1961 *Phys. Rev.* **124** 925–35
Halford W D 1972 *J. Math. Phys.* **13** 1699–703
Mahanta M N and Reddy D R K 1971 *J. Math. Phys.* **12** 929–32
—— 1972 *J. Math. Phys.* **13** 708–9
—— 1974 *J. Math. Phys.* **15** 1235–37
Patel L K 1975 *Tensor, New Ser.* **29** 237–40
Reddy D R K 1976 *Ind. J. Pure Appl. Math.* in the press
Sen D K and Dunn K A 1971 *J. Math. Phys.* **12** 578–88
Taub A H 1951 *Ann. Math., NY* **53** 472–90